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**Stochastic Tastes and Money
in a Neo-Keynesian Economy**

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Abstract

This paper studies a monetary economy with heterogeneous agents in which trade takes place in a centralized market. Each agent is a potential producer and consumer of a service (or perishable good) but has stochastic preferences that determines his taste for the good in each period in time. Money serves as a medium of exchange as well as a store of value. We prove existence of stationary fix-price equilibria with exogenously given quantity of money in which transactions can take place at non-Walrasian prices. Precautionary savings, under- and oversupply, dynamics on money holdings, and the effects of changes in the quantity of money are discussed.

JEL-Classification: D51, E41, E52.

Keywords: Microeconomic Models of Money, Stochastic Preferences, Fix-Price Equilibria, Rationing, Dynamics of Money Distributions.

1 Introduction

Trade of standardized goods takes place mainly in well-organized markets. This point of view is common to most models in general equilibrium theory. In recent search-theoretic models of money—pioneered (in an apparently forgotten paper) by Hellwig (1976)—however, the focus is exclusively on trade in bilateral meetings in which pairs of agents are assigned randomly (Diamond 1982, 1984), (Kiyotaki and Wright 1991, 1993), (Wallace 1998, and the references therein). While the search-theoretic approach has improved our understanding of the role of money as a medium of exchange as well as the real effect of changes in the quantity of money, it does not provide an appropriate framework for the understanding of monetary economies in which there is one (centralized) market for each good. Hellwig (1993) has pointed out the difficulties associated to such a centralized-market approach where bilateral bargaining does not take place.

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In the model discussed in this paper there is one non-storable consumption good (or service) that can be produced and consumed by every agent. All demanders and suppliers of the good meet in a spot market for that particular good in each period in time. Trade takes place as an exchange of money against the good. Each agent's decision which market side to choose is endogenous and depends on realized individual stochastic preferences as well as on his money holdings in the respective period. Agents are heterogeneous with respect to their time-preference and possible tastes. Stochastic preferences of agents have far-reaching consequences. The fact that agents are uncertain about their future preference for the good implies that a direct exchange of goods would require infinitely many markets for contingent commodities. The approach is related to neo-Keynesian disequilibrium models due to the following two assumptions. First, following the fix-price literature (Benassy 1982, Drèze 1991, Malinvaud 1977), transactions take place at non-Walrasian prices, where prices are exogenously given in the model. Second, it is assumed that the good is non-divisible such that the long market side is rationed by a zero-one law. Rationing probabilities are endogenously determined.

The basic modelling-approach that is advanced in the model discussed here has first been suggested and experimentally tested in Hens, Schenk-Hoppé, and Vogt (2001). The purpose of the present paper is to make rigorous the intuitive reasoning provided there and to extend the model by allowing heterogeneity of agents and non-trivial stochastic preferences. Recall that in Hens, Schenk-Hoppé, and Vogt (2001) all agents are assumed to be alike (i.e. a representative agent framework can be adopted because of the lack of heterogeneity among agents) and the stochastic preferences can only take on two values. The experimental findings in Hens, Schenk-Hoppé, and Vogt (2001) give strong support to the modelling-approach as well as to the advanced version presented here. In particular their results confirm the appropriateness of the assumption of agents' individual rationality and agents heterogeneity.

The model and the assumptions can both be motivated by trade circles. Based on the labor theory of value (Ricardo 1817), there is essentially only one good (labor) that is traded in a centralized market at a fixed price and in fixed quantities. An example is provided by the Great Capitol Hill Baby-sitting Coop (Sweeney and Sweeney 1977) that has been popularized by Krugman (1999).

The model allows for a detailed analysis of effective demand and precautionary savings in (non-Walrasian) stationary equilibria of a centralized-market economy. We further study the dynamics on agents' money holdings and the effect of (exogenous) changes in price and nominal quantity of money on stationary equilibria. Some conclusions for monetary policy can be drawn from the model.

The model is explained in Section 2. The analysis of individual behavior and existence of stationary market equilibria is presented in Sections 3.1 and 3.2, respectively. Section 3.3 discusses monetary policy. Section 4 concludes. All proofs are relegated to the Appendix.

2 Model

The economy consists of a continuum of infinitely-lived agents, denoted by the interval $[0, 1]$. Time is discrete and denoted by $t \in \mathbb{N}$. There is one non-storable commodity, e.g. labor. Every agent can either demand or supply one unit of the good in each period in time. Production and consumption need both one period in time and, thus, an agent can either consume or produce the good but cannot consume his own good. Production incurs no costs. There is a fixed—exogenously given—quantity of a tangible non-consumable good, called money. We assume that agents have unknown histories (hence money is essential).

A spot market for the good opens in each period—the price being exogenously fixed. Every agent has to decide whether to stay out of the market, demand, or supply the good. Transactions take place as an (anonymous) exchange of one unit of the good against money. Agents face a cash-in-advance constraint, i.e. buying is only possible with money. Private credit is ruled out by the assumption of anonymity of agents. Rationing takes place as a zero-one law.

Agents are heterogeneous with respect to their time-preference and taste. We assume that there are only finitely many different types of agents, denoted by $i \in I$. The agents of each type form a measurable subset of $[0, 1]$ with strictly positive Lebesgue-measure λ^i , $i \in I$ ($\sum_i \lambda^i = 1$). The characteristics of all agents of type i are as follows. Time-preference is determined by a discount rate $0 < \beta^i < 1$. Every agent has stochastic preferences for the good that vary over time. In any period in time his taste for the good (that is, the instantaneous utility from consumption) in the respective period is determined by an independent and identically distributed random draw¹. Formally, let the finite set U^i denote the possible realizations of an i.i.d. stochastic process $(u_t^i)_{t \geq 0}$ with stationary probability distribution p^i . We assume that $\min U^i > 0$, and $p_u^i = p^i(u) > 0$ for all $u \in U^i$. There is uncertainty with respect to their future taste if U^i has at least two distinct elements. At the beginning of each period every agent privately observes the revealed value of his current-period taste before making a decision.

Each agent faces an intertemporal optimization problem for perceived constraints on the market: the demand-supply decision in every period subject to his revealed taste and money holdings. Strategic behavior is ruled out by the assumption of a continuum of agents. Thus an agent does not take into account the possibility of the population to influence market-averages by collateral deviations from individually rational decisions. Because we will restrict our analysis to stationary equilibria, we consider the agent's maximization problem in a stationary market. That is, the price of the good is constant over time and initial money holdings is a multiple of the price. Normalization allows us to fix the price at one and to let money holdings $m_t \in \mathbb{N} = \{0, 1, 2, \dots\}$ be integers. We also suppose that agents face constant success probabilities $p_B > 0$ and $p_S > 0$.

¹The taste for the good today does thus not convey any information on his taste in any future period.

These probabilities are determined by the rationing scheme and are endogenous to the model. p_B denotes the probability that an agent can actually purchase if demanding the good (B = buy). Analogously p_S is the probability that an agent actually sells if supplying the good (S = sell).

We consider a representative agent of a particular type i , omitting the index of the type to simplify notation.

The state variable is a pair $(m_t, u_t) \in S := \mathbb{N} \times U$ of current money holdings and revealed taste. Money holdings is an endogenous variable, while the taste is exogenously determined.

The set of feasible actions depends on the beginning-of-period money holdings m , formalized by the correspondence $\Gamma : S \rightarrow A := \{B, S, O\}$ where (B = buy), (S = sell), and (O = out of market). For all $u \in U$, $\Gamma(0, u) = \{S, O\}$ (the credit constraint prevents agents without money to demand the good), and $\Gamma(m, u) = A$ for all $m \geq 1$. The graph of Γ is the (measurable) set $D = \{(m, u, a) \mid a \in \Gamma(m, u)\} \subset \mathbb{N} \times U \times A$.

We assume rationing is i.i.d. across time as well as across the two actions B (buy) and S (sell). Formally, let $z_t = (z_t^1, z_t^2)$, $t \geq 0$, be a stochastic process on $Z = \{0, 1\}^2$ such that z_t^1 and z_t^2 are i.i.d. processes with distribution $(1 - p_B, p_B)$ resp. $(1 - p_S, p_S)$, and z_s^1 and z_t^2 are independent for all $s, t \in \mathbb{N}$. $z^1 = 1$ resp. $z^2 = 1$ means that the agents buy resp. sell. How to implement the rationing scheme is detailed at the end of this section.

The evolution of the agent's money holdings over time is determined by the chosen action and on the outcome of the rationing process. Fix a period in time t . Given beginning-of-period data (money holdings m , realized preference value u , and chosen action a), the new, i.e. beginning-of-next-period, money holdings and the instantaneous utility received in the current period are determined by a function $H : D \times Z \rightarrow \mathbb{N}$ resp. $F : D \times Z \rightarrow U$, where

$$\begin{aligned} H(m, u, B, z) &= m - 1, & \text{and} & & F(m, u, B, z) &= u, & \text{if } z_1 &= 1; \\ H(m, u, S, z) &= m + 1, & \text{and} & & F(m, u, S, z) &= 0, & \text{if } z_2 &= 1; \\ H(m, u, a, z) &= m, & \text{and} & & F(m, u, a, z) &= 0, & \text{otherwise.} \end{aligned}$$

For instance, if the agent demands the good and he is not rationed then his money holdings decrease by one and he receives the instantaneous utility corresponding to his current preference value. Staying out of the market or being rationed implies no change in the money holdings and an instantaneous utility of zero.

Let u^t resp. z^t denote the revealed preferences resp. history of the rationing outcome up to time t with corresponding space $(U^t, \mathcal{B}(U^t))$ resp. $(Z^t, \mathcal{B}(Z^t))$.

A feasible plan from $s_0 = (m_0, u_0) \in S$ is a sequence of functions $\pi = \{\pi_t\}_{t=0}^\infty$ with $\pi_t : U^t \times Z^t \rightarrow A$ being $\mathcal{B}(U^t) \otimes \mathcal{B}(Z^t)$ -measurable for all $t \geq 0$ such that $\pi_0 \in \Gamma(s_0)$ and $\pi_t(u^t, z^t) \in \Gamma(m_t^\pi(u^{t-1}, z^t), u_t)$ for all $(u^t, z^t) \in U^t \times Z^t$, $t \geq 1$, where the functions $m_t^\pi : U^{t-1} \times Z^t \rightarrow \mathbb{N}$ (the agent's money holdings at the beginning of period t) are defined recursively by $m_1^\pi(u_0, z) = H(m_0, u_0, \pi_0, z)$ for all $z \in Z$, and $m_{t+1}^\pi(u^t, z^{t+1}) = H(m_t^\pi(u^{t-1}, z^t), u_t, \pi_t(u^t, z^t), z_{t+1})$ for all $(u^t, z^{t+1}) \in U^t \times Z^{t+1}$, $t \geq 1$. Denote by $\Pi(s_0)$ the set of feasible plans from s_0 . For each $\pi \in \Pi(s_0)$, define $m^\pi = \{m_t^\pi\}_{t=1}^\infty$ as above. Further, let $m_0^\pi = m_0$.

Given a discount rate $\beta \in (0, 1)$, for any initial state $s_0 = (m_0, u_0)$ the agent determines a plan that takes on the supremum

$$v^*(s_0) := \sup_{\pi \in \Pi(s_0)} \mathbb{E} \left(\sum_{t=0}^{\infty} \beta^t F(m_t^\pi, u_t, \pi_t, z_{t+1}) \right) \quad (1)$$

Assuming risk-neutrality of agents, we can restrict the analysis to the expected value of $v^*(s_0)$ with respect to the realization of u_0 ,

$$V^*(m) = \sum_{u \in U} p_u v^*(m, u) \quad (2)$$

$V^*(m)$ is the maximum expected discounted utility an agent can achieve when initially endowed with m units of money.

By the principle of optimality, $V^* : \mathbb{N} \rightarrow \mathbb{R}$ solves the Bellman equation,

$$\begin{aligned} V(0) &= \beta \max \left\{ p_S V(1) + (1 - p_S) V(0), V(0) \right\} \\ V(m) &= \beta \sum_{u \in U} p_u \max \left\{ p_B [u/\beta + V(m-1)] + (1 - p_B) V(m), \right. \\ &\quad \left. p_S V(m+1) + (1 - p_S) V(m), V(m) \right\} \end{aligned} \quad (3)$$

where the decision in the maximization operator is contingent on the realization u of the stochastic preferences. Equivalence of the Bellman equation and the corresponding sequence problem follows from validity of the transversality condition which holds because the return function is bounded.

Denote the solution to the Bellman equation by V^* —existence and uniqueness is proved in the next section. $V^*(m)$ is the indirect utility of an agent holding m units of money. The decision in (3) defines an optimal stationary policy $\psi^* : \mathbb{N} \times U \rightarrow \{B, S, O\}$.

The agents' optimal stationary policies give rise to actual success probabilities as follows. The market coordinates demand and supply of all agents—who follow their individually optimal policies $\psi_i^* : \mathbb{N} \times U^i \rightarrow \{B, S, O\}$. Since the price of the good is fixed, demand and supply are coordinated by a zero-one rationing scheme as described above. Denote by $\mu_i(m)$ the distribution of agents over money holdings and types— $\mu_i(m)$ is the Lebesgue-measure of agents of type i holding m units of money.

Given policies ψ and a distribution μ , the total supply of the good is given by

$$\mathbf{S}(\psi) = \mathbf{S}(\psi_{i \in I}) = \sum_{m \in \mathbb{N}, i \in I} \mu_i(m) \sum_{u \in U^i} p_u^i \mathbf{1}_{\psi_i(m, u)}(S)$$

and the total demand can be equated as

$$\mathbf{D}(\psi) = \mathbf{D}(\psi_{i \in I}) = \sum_{m \in \mathbb{N}, i \in I} \mu_i(m) \sum_{u \in U^i} p_u^i \mathbf{1}_{\psi_i(m, u)}(B)$$

The actual success probabilities p_S and p_B are therefore given by

$$p_B = \min \left\{ 1, \frac{\mathbf{S}(\psi)}{\mathbf{D}(\psi)} \right\}, \quad p_S = \min \left\{ 1, \frac{\mathbf{D}(\psi)}{\mathbf{S}(\psi)} \right\} \quad (4)$$

provided the rationing scheme selects the same relative amount of agents for each type and money holdings, i.e. the fraction p_B (p_S) of agents demanding (supplying) the good is selected to trade in every state (i, m) . As a convention both probabilities are set to zero if there is either no demand or no supply. We postpone a detailed description how such a rationing scheme can be implemented to the end of this section. It may suffice for the moment to note that such a scheme exists.

Let us explain the possible non-tâtonnement dynamics of the model before defining a stationary equilibrium. Let the price be fixed and publicly announced by some outside institution. Suppose there are given an initial distribution of (nominal) units of money over agents (such that all individuals have integer real money holdings) and individually presumed (given by birth, say) success probabilities. Each agent chooses a policy (i.e. a function that determines the action contingent on money holdings and instantaneous utility) that is optimal (i.e. yields highest discounted expected utility) for the perceived market conditions (i.e. publicly announced price and individually perceived success probabilities). After the individual state of preference for the good is revealed for all agents, each agent announces whether he demands or supplies the good or stays out of the market in the current period—this decision is determined by the optimal policy. The individual decisions imply a total demand and supply for the good at the fixed price in the respective period in time. According to the above rationing scheme, success probabilities for the agents on both sides of the market are determined and publicly announced. Each agent uses this information to update the individually perceived market conditions for the next period. One would need to specify this procedure to arrive at a full model. Then the rationing mechanism is put to work and assigns to each agent on the shorter side of the market an (anonymous) agent on the other side. Transactions are carried out at the fixed price, which leads to a change in the distribution of agents of each type over money holdings. A new period begins in which the economy evolves according to the process of decision-making and market-interaction just described.

Detailing this dynamics and giving a convincing explanation of the specific assumptions is a formidable task. A suitable framework for such an approach is provided by theory of random dynamical systems (Arnold 1998); though it is certainly challenging to derive analytical results. See e.g. Schenk-Hoppé (2001) for some economic applications.

Our aim in this paper is more modest. We focus exclusively on stationary equilibria which can be regarded as states of rest of the dynamics outlined above. In a stationary equilibrium, (1) perceived and actual rationing probability coincide, (2) the policy of each agent is individually optimal for given market conditions, and (3) the distribution of money is stationary. Thus agents have no incentive to change their perceived data or to deviate from their policy. This in turn leads to stationary market-conditions. A formal definition is as follows, extending the notion introduced in Hens, Schenk-Hoppé, and Vogt (2001).

Definition 1 A stationary equilibrium $(\psi^*, \mu^*, p_B^*, p_S^*, M^*)$ is a list of stationary policies $\psi^* = \psi_{i \in I}^*$, a probability measure μ^* of agents over money holdings and types, success probabilities p_B^* and p_S^* , and a real quantity of money M^* , such that

- (i) given p_B^* and p_S^* , ψ_i^* is an optimal policy for the agents of type i for all $i \in I$, i.e. ψ_i^* is a stationary policy in (3);
- (ii) given ψ^* , p_B^* and p_S^* , μ^* is a stationary measure for the corresponding Markov chain on $I \times \mathbb{N}$ with $\sum_{m \in \mathbb{N}} \mu_i^*(m) = \lambda^i$ for all $i \in I$ (λ^i is the Lebesgue-measure of agents of type i);
- (iii) given ψ^* and μ^* , the probabilities p_B^* and p_S^* satisfy (4);
- (iv) the average money holdings is equal to the average money supply, i.e. $\sum_{m \in \mathbb{N}, i \in I} m \mu_i^*(m) = M^*$.

A stationary equilibrium is called *monetary*, if the value function solving (3) is strictly positive on \mathbb{N} for at least one type of agents.

We close this section with a detailed description of the implementation of the stochastic preferences, rationing scheme, and matching for trade.

In the assignment of preference values the problem is to assign values $U = \{u_1, \dots, u_n\}$ to a continuum of agents $[0, 1[$ (after a homeomorphic transformation of the set of agents) such that (1) each agent faces an i.i.d. process with distribution $p = (p_1, \dots, p_n)$, and (2) the fraction of agents with value u_j has Lebesgue-measure p_j for each realization of the assignment process.

Let ξ be an i.i.d. random variable uniformly distributed on $[0, 1[$. Define $p_0 := 0$. For each realization of ξ , we assign to all agents in the interval $[\xi + \sum_{0 \leq k < j} p_k, \xi + \sum_{0 \leq k \leq j} p_k[\mod 1 = (\xi + \sum_{0 \leq k < j} p_k + [0, p_j]) \mod 1$ the value u_j , $j = 1, \dots, n$. This assignment mechanism has all required properties.

The rationing scheme can be implemented completely analogous. Assume demanders are rationed for instance. Fix a group of agents being of the same type and having identical money holdings. Let $U = \{0, 1\}$ and $p = (1 - p_B, p_B)$. The above mechanism assigns the value 1 (represents that an agent can actually trade and buy the good) to the fraction p_B of agents. And each demander faces an i.i.d. random draw that selects him for trade with probability p_B . We can even assume independence of the rationing scheme over types and money holdings.

We finally describe how to implement a matching mechanism for the assignment of trading partners. Line up all intervals of agents who purchase and all agents who sell the good. One ends up with two intervals of the same length, for convenience normalized to $[0, 1[$. The members of either group have been selected by the above rationing scheme, if both market sides do not match. Let η be the realization of an i.i.d. random variable with uniform distribution on $]0, 1[$. Assign buyer x to seller $\eta + x \mod 1$, $x \in [0, 1[$.

3 Analysis

In the analysis of the monetary economy we proceed in two steps. We first study the optimal individual behavior of agents for given success probabilities. Afterwards we look into the problem whether the individual decisions can be made compatible in the market framework introduced in the previous section.

3.1 Individual Behavior

The optimal individual behavior of every agent is described—though only implicitly—by the Bellman equation (3). Our goal in this section is to provide a full (and as explicit as possible) description of the optimal stationary policy. We consider an agent of a fixed type (thus omitting the index) with discount rate $0 < \beta < 1$ and preferences $u \in U$ with associated probabilities $p = (p_u)$. Throughout the following we assume that perceived success probabilities are strictly positive, $p_B > 0$ and $p_S > 0$. (Other stationary equilibria are simple to understand.)

We first present an existence and uniqueness result for the solution to the Bellman equation (3).

Lemma 1 *There exists a unique solution V^* to the Bellman equation (3). V^* is strictly positive, strictly increasing, bounded, and strictly concave.*

Lemma 1 ensures that the indirect utility of m units of money, $V^*(m)$, is always strictly positive, i.e. money has value if transactions take place with some strictly positive probability ($p_B > 0$, $p_S > 0$). In particular staying out of the market cannot be optimal for an agent; we therefore can exclude this option in all further considerations. Agents' marginal utility from possessing an additional unit of money is diminishing and—because the value function is bounded—vanishing in the limit.

The following auxiliary result on the optimal stationary policy is straightforward from the objective function of the agent.

Lemma 2 *For all $m \geq 1$, $\psi^*(m, \max U) = B$.*

These qualitative properties enable us to completely specify the optimal policy.

Theorem 1 *The optimal policy is given by $\psi^*(m, u) = S$ for all $0 \leq m < m_u$ and $\psi^*(m, u) = B$ for all $m \geq m_u$, $u \in U$, where*

$$m_u := \min \left\{ m \geq 1 \mid \frac{p_B u}{\beta} > p_S [V^*(m+1) - V^*(m)] + p_B [V^*(m) - V^*(m-1)] \right\}$$

The quantity of money m_u is uniquely related to the opportunity costs of accumulating another unit of money. Since the right-hand side in the defining equation for m_u is a (strictly) decreasing function of m , m_u decreases as u

increases. Thus opportunity costs are increasing in the potential instantaneous utility u and in the money holdings. These considerations also make clear that $m_{\min U}$ is the maximal amount of money an agent wants to hold. Obviously trade will break down in an economy with a real quantity of money exceeding this maximal amount for all types.

Agents with stochastic preferences (i.e. $\min U < \max U$) can have precautionary savings. This is immediate from Theorem 1. Suppose $m_u = 1$ if and only if $u = \max U$. Then an agent holding one unit of money only demands the good if his highest instantaneous utility is revealed. For all other utility-values he supplies the good. This follows from the diminishing marginal utility from money. The values of the instantaneous utility that an agent sacrifices in order to keep his precautionary savings are the lower the more money he holds. In fact, let $1 < m_u < m_{\min U}$ for some $u \in U$. Then an agent with $m = m_u$ demands the good (i.e. plans to spend money) if the revealed instantaneous utility u_t is higher than or equal to u . The agent keeps his money, if the revealed taste is too low compared to the value assigned to his money holdings m_u —hence the agent puts aside precautionary savings for the case that a sequence of high utility-values may occur.

Let us briefly discuss the deterministic case $U = \{u\}$ with $u > 0$. The optimal policy is given by $\psi^*(0, u) = S$ and $\psi^*(m, u) = B$ for all $m \geq 1$. This is immediate from Lemma 2 that states $m_{\max U} = 1$. An agent with deterministic preferences will demand the good when holding money and supply whenever he is without money. He voluntarily sets his maximal money holdings to one. This finding provides a rational for the one unit storage capacity imposed by (Kiyotaki and Wright 1991, 1993) and (Wallace 1997), where agents' maximal money holdings are exogenously limited to one.

If the taste can take on two distinct values, the model has certain similarities with a recent search-theoretic model by Berentsen (2000) in which the above restriction on the storage capacity is relaxed and the upper bound on money holdings is endogenous.

Finally let us note that the optimal policy is not necessarily unique because an agent can be indifferent between demanding and supplying the good. This situation occurs if the condition in the defining equation of m_u holds with equality for $m = m_u - 1$. An agent holding $m_u - 1$ units of money is indifferent between B and S when u is revealed.

3.2 Stationary Equilibria

Having characterized the optimal individual behavior for given market conditions, we next prove existence of stationary equilibria in which the individual decisions are compatible.

Theorem 2 *Given an economy $(\beta^i, U^i, p^i, \lambda^i)_{i \in I}$ and any prescribed success probabilities $p_B > 0$ and $p_S > 0$ with $\max\{p_B, p_S\} = 1$. Then there exists a monetary stationary equilibrium with $p_B^* = p_B$, $p_S^* = p_S$.*

This result ensures existence of monetary stationary equilibria provided that the quantity of money can be chosen by an institution outside the economy. The construction of a stationary distribution over money holdings for given characteristics of agents is the essential part in the proof. Starting from the prescribed success probabilities, the optimal policy of each agent can be determined. Then a corresponding (irreducible) Markov chain on the set of money holdings can be defined for each type. The stationary distribution for the economy is then constructed from these type-specific stationary distributions. The conditions for a monetary stationary equilibrium are satisfied by construction of this stationary distribution, roughly speaking.

An immediate but important consequence of Theorem 2 is the existence of a real quantity of money such that no agent is rationed. (Corollary 1 is a restatement of Theorem 2 for $p_B = p_S = 1$.)

Corollary 1 *In any economy there exists a Walrasian equilibrium, i.e. there is a monetary stationary equilibrium such that $p_B^* = p_S^* = 1$.*

Obviously the number of trades is maximal in a Walrasian equilibrium. It is justified, from this perspective, to call the corresponding real quantity of money the *optimum quantity of money* in the economy.

In a monetary stationary equilibrium the real quantity of money is given by $M^* = \sum_{m \in \mathbb{N}, i \in I} m \mu_i^*(m)$ (viz. the average-per-capita real amount of money), where μ^* is the stationary distribution of money.

Every quantity of money being different to M^* leads to a monetary stationary equilibrium with strictly less trades than with M^* as long as the money holdings m_u^i (cf. Theorem 1) do not change for all $i \in I$, $u \in U^i$. This can be seen as follows. Let M^* be the optimum quantity of money. Since all m_u^i are fixed under small variations in the quantity of money around M^* , in any monetary equilibrium with a higher resp. lower quantity of money than M^* , the success probability of buying resp. selling is strictly less than one (buyers resp. sellers are rationed). For instance an increase in the real quantity of money entails more buyers in the market as long as all agents retain their policy because the distribution of money moves toward higher money holdings (as is clear from the construction of the Markov chain on the set of money holdings in the proof of Theorem 2). In a stationary equilibrium the implicit price of the good is typically higher than the equilibrium price due to rationing.

Let us briefly address the existence of equilibria in which money is not valued. A non-monetary equilibrium is given by $p_B^* = 0$ and a distribution of money that assigns at least one unit of money to every agent. Individual rationality suggest that selling is of no use because the money gained from trade cannot be spend in any future period in time due to lack of supply. All agents would stay out of the market which implies no supply in this equilibrium. If money endowments are sufficiently high, then this equilibrium is robust.

3.3 Monetary Policy

There are two mechanisms to move an inefficient equilibrium towards an efficient one. The real quantity of money $M^* = \bar{M}/p$ is the ratio of nominal quantity of money \bar{M} and (fixed) price p . Both present instruments to vary the real quantity of money and can be made endogenous to the model.

On the one hand, changes in the price can be triggered by those agents who face the risk of being rationed because they have an incentive to trade the good at a price that is different from the given one. On the other hand, a benevolent social planner who observes effective demand and supply can issue or discard units of money to achieve the optimum quantity of money. For instance in a sellers' market agents would like to offer or ask the good at a higher price, and a planner would be inclined to discard money in order to move from an equilibrium with too high real quantity of money to a Walrasian equilibrium.

Incorporating these mechanisms in the model to arrive at a fully-specified endogenous model is a challenging task, and it is much beyond the scope of this paper. Here we resort to comparative statics in discussing monetary policy implications of the model. We compare the properties of different equilibria and examine the willingness of agents to agree to a transition.

It is clear from the agent's maximization problem (1) that the value function V^* is maximal for given money holdings m if and only if $p_B^* = p_S^* = 1$. From an individual perspective the implementation of the Walrasian equilibrium is unambiguously preferred as long as the agent's real money balances are not decreased.

Moving from a buyers' market, i.e. an equilibrium in which there is over-supply, to a Walrasian equilibrium is preferred by every agent in the economy. Any such transition requires a decrease in the price or an increase in the nominal quantity of money because a higher real quantity of money is needed when moving from a buyers' market to a Walrasian equilibrium. The impact of either change is welcomed by all agents—because real money holdings do not decrease—and the economy eventually is in a Walrasian equilibrium.

The situation is different if there is undersupply in a monetary equilibrium. In a sellers' market, i.e. in an equilibrium with a too high real quantity of money, the transition to a Walrasian equilibrium requires an increase in the price or the annihilation of nominal units of money. For each agent there is a trade-off between the gain in expected discounted utility ($V^*(m)$ is increasing for fixed m when $\min\{p_B, p_S\}$ tends to 1) and the loss in real money holdings (real money holdings m decreases when the real quantity of money decreases). Only an agent with no money unambiguously favors the transition to a Walrasian equilibrium. All other agents can be reluctant to agree to a change in the real quantity of money. We can conclude that involuntary unemployment in an undersupply equilibrium is possible in the monetary-economy model.

Let us finally discuss how money can be issued or discarded in the economy. Any change in the quantity of money leads to an equilibrium with a different stationary distribution of agents over types and money holdings. A mechanism that immediately leads from one equilibrium to another would need a one-time

taxation (or cash allowance) that depends on the type as well as on the money holdings of agents. However, such a distribution mechanism is not necessary to accomplish this transition.

In fact it does not matter how money is given away (or collected) as long as agents' beliefs about the rationing probabilities p_B and p_S are appropriately updated. Suppose the rationing probabilities corresponding to the equilibrium associated to the new quantity of money are publicly announced. If all agents act according to the individually optimal policy, given these data, then the distribution of money converges toward the unique stationary distribution associated to the equilibrium. This follows from the fact that the Markov process on the set of types and money holdings is irreducible. Hence any initial distribution generating the exogenously fixed quantity of money converges to the unique stationary distribution if agents perceived rationing probabilities are the right ones. However, if agents' perceptions of the market conditions are not fixed but vary over time, the transition to the new stationary equilibrium may be impossible and cyclic or other complex dynamics might occur. Detailing this learning procedure and studying the resulting (non-trivial) dynamics is an interesting task for future research.

4 Conclusions

This paper discusses a microeconomic model of a neo-Keynesian monetary economy, advancing the approach presented in Hens, Schenk-Hoppé, and Vogt (2001). The approach complements recent search-theoretic models of money. In the economy there is one good that is traded in a centralized market by heterogeneous agents. The adjustment of the price is sluggish, and quantity-adjustments are impossible due to the indivisibility of the good. Agents can produce and consume the good but only one action is possible in every period. Their preference for the good is stochastic and only the individual taste in the current period is revealed to each agent. Agents' demand-supply decisions are based on individual current tastes and money holdings. Money has two roles in this economy: it serves as a medium of exchange as well as a store of value. The cash-in-advance constraint for all agents, i.e. the absence of private credit, is important in the model. Money in the model is exclusively outside money. The results would not change, however, if agents face a fixed credit limit.

We define a stationary equilibrium for this economy in which transactions take place at non-Walrasian prices. Agents are individually rational but take market-conditions as given. This approach in particular captures the short-run rigidity of prices. Market clearing is ensured by a zero-one rationing scheme. The notion of an equilibrium is particularly justified if agents give relatively little weight to consumption in the far future.

Equilibria in which money is valued exist in any population of heterogeneous agents if the amount of money can be exogenously determined. The properties of different equilibria and the transition to Walrasian equilibria are discussed. We show that oversupply equilibria are not robust, but involuntary unemployment

can prevail in undersupply equilibria.

We outline some possible improvements of the model, which we believe to be particularly suitable for future research. Monetary policy can be made endogenous to the model by detailing the adjustment processes for prices or the quantity of money. We also discuss the possibility to describe a non-tâtonnement dynamics for the update of agents' appraisal of market conditions such that perceived rationing probabilities vary if the economy is not in a stationary equilibrium. Open-market operations of a central bank could also be incorporated into the model by allowing agents to borrow at a (non-zero) interest rate.

A Appendix

Proof of Lemma 1. We first prove existence and uniqueness of the solution to the Bellman equation. Denote by X the set of bounded functions $V : \mathbb{N} \rightarrow \mathbb{R}$. Let $\bar{u} := \max\{u \mid u \in U\}$. The (non-empty) subset of X

$$Y := \{V : \mathbb{N} \rightarrow \mathbb{R} \mid 0 \leq V(m) \leq \bar{u}/(1 - \beta) \text{ for all } m \in \mathbb{N}\}$$

is complete with respect to the supremum norm. We define an operator $T : X \rightarrow X$ via the Bellman equation (3) as follows. For each function $V \in X$, define the image TV by,

$$\begin{aligned} TV(0) &:= \beta \max \left\{ p_S V(1) + (1 - p_S) V(0), V(0) \right\} \\ TV(m) &:= \beta \sum_{u \in U} p_u \max \left\{ p_B [u/\beta + V(m-1)] + (1 - p_B) V(m), \right. \\ &\quad \left. p_S V(m+1) + (1 - p_S) V(m), V(m) \right\} \end{aligned}$$

Obviously, any solution to the Bellman equation is a fixed point of T and vice versa. We show that (i) T is a contraction on X ; and (ii) T maps Y into itself. This implies that there exists a fixed point of T in Y which is unique in X .

(i): Blackwell's sufficient conditions for a contraction, see e.g. Stokey, Lucas, and Prescott (1989, Theorem 3.3), says that T is a contraction on X , if T is (1) monotone (if $V, W \in X$ with $V(n) \leq W(n)$ for all $n \in \mathbb{N}$ then $TV(n) \leq TW(n)$ for all $n \in \mathbb{N}$), and (2) discounting (there exists some constant $\gamma \in (0, 1)$ such that $T(V + c)(n) \leq TV(n) + \gamma c$ for all $V \in X$ and $c \geq 0$, where the sum of $V \in X$ and a constant function is defined by $(V + c)(n) := V(n) + c$). It is straightforward to check monotonicity and discounting of the operator T defined above.

(ii): $V(m) \leq \sum_{t=0}^{\infty} \beta^t \bar{u} = \bar{u}/(1 - \beta)$ for any policy. Since T is monotone and maps the set of positive functions into itself (because $u \geq 0$ for all $u \in U$), it suffices to show that the image of the constant function $W(n) \equiv \bar{u}/(1 - \beta)$ satisfies $TW(n) \leq \bar{u}/(1 - \beta)$ for all $n \geq 0$. The definition of T yields that $TW(n) \leq p_B \bar{u} + \beta \bar{u}/(1 - \beta) \leq \bar{u}/(1 - \beta)$, for all $n \geq 0$. This yields invariance of Y .

The proof of (ii) also yields boundedness of V^* .

V^* is strictly positive: For all $m \geq 1$, $V^*(m) \geq p_B \mathbb{E}u + \beta V^*(m-1) \geq p_B \mathbb{E}u > 0$ because V^* is non-negative by the proof of (ii). In particular we have $V^*(1) > 0$ and hence $V^*(0) \geq \beta p_S V^*(1) > 0$. This implies $V^*(m) > 0$ for all $m \geq 0$.

V^* is strictly increasing: The set of feasible plans is increasing in the initial endowment of money m_0 and, therefore, $V^*(m+1) \geq V^*(m)$ for all $m \geq 0$. Moreover, the actual payoff for any elementary event, i.e. along any sample path of the exogenous stochastic processes, is increasing in the initial endowment of money, because the sequence of decisions made by an agent with endowment m can also be carried out if the endowment is $m+1$. To prove the assertion, it therefore suffices to show that for an event with strictly positive probability the total utility is strictly increasing in the initial endowment of money. Suppose a run of m -times u_n occurs and the agent actually gets the opportunity to buy the good in all periods in time (an event that has strictly positive probability for each fixed $m \geq 1$). According to Lemma 2, at the end of this run an agent with initial endowment $m+1$ holds 1 unit of money, whereas an agent with endowment m is left with no money. The difference in the value of the two positions is $V^*(1) - V^*(0)$ at the end of the run. As noted above, $V^*(0) > 0$. This implies $V^*(1) - V^*(0) = (1 - \beta)/(1 - \beta + \beta p_S) V^*(0) > 0$.

Finally, we show that V^* is strictly concave. Denote by ψ^* the corresponding optimal policy. Suppose V^* is not strictly concave, i.e. there exists an $\tilde{m} > 0$ such that $V^*(\tilde{m}+1) - V^*(\tilde{m}) \geq V^*(\tilde{m}) - V^*(\tilde{m}-1)$.

By Lemma 2 $\psi^*(\tilde{m}+1, \bar{u}) = B$. The Bellman equation (3) implies

$$V^*(\tilde{m}+1) = \frac{\beta}{1-\beta} \left[p_{\bar{u}} p_B \left(\frac{\bar{u}}{\beta} + V^*(\tilde{m}) - V^*(\tilde{m}+1) \right) + \dots \right]$$

where the remaining terms on the far right of this equation is determined by the (unknown) optimal policy $\psi^*(\tilde{m}+1, u)$, $u \in U$, $u \neq \bar{u}$.

Suppose \bar{u} is revealed and the agent actually buys (which happens with positive probability). Then an agent who pursues from thereon the optimal policy corresponding to one unit of money less than she actually possesses would not be worse off because $V^*(\tilde{m}+1)$ is not decreased. This is straightforward from the assumption that V^* is not concave at \tilde{m} . However, this contradicts optimality of the policy, because disposing one unit of money leads to a strictly lower total utility because V^* is strictly increasing in the money holdings as has been proved above. \square

Proof of Lemma 2. $\bar{u} := \max\{u \mid u \in U\} > 0$ is the maximal value of the return function F in the agent's objective function (1). Since the success probabilities p_B and p_S are independent of time and $0 < \beta < 1$, postponing the action B when \bar{u} is observed and $m \geq 1$, strictly decreases total utility. \square

Proof of Theorem 1. We need to show that $\psi^*(m, u) = S$ for all $0 < m < m_u$ and $\psi^*(m, u) = B$ for all $m \geq m_u$. $\psi^*(0, u) \equiv S$ holds by Lemma 2.

Consider the Bellman equation (3). An agent (weakly) prefers to sell when u is revealed, if $p_B (u/\beta + V^*(m-1)) + (1 - p_B) V^*(m) \leq p_S V^*(m+1) + (1 - p_S) V^*(m)$. If the inequality is reversed, he prefers to buy when observing u .

The condition is equivalent to $p_B u/\beta \leq p_B(V^*(m) - V^*(m-1)) + p_S(V^*(m+1) - V^*(m))$.

$V^*(m) - V^*(m-1)$ is strictly decreasing in m because V^* is strictly concave by Lemma 1. Lemma 1 also ensures $\lim_{m \rightarrow \infty} V^*(m) - V^*(m-1) = 0$ because $V^*(m)$ is increasing and uniformly bounded, i.e. $\bar{V} := \lim_{m \rightarrow \infty} V^*(m)$ exists and is finite.

Summarizing these findings—recalling the above characterization of the optimal policy— m_u is a well-defined finite integer number and the policy stated in the theorem is actually optimal. \square

Proof of Theorem 2. The proof relies on the results of Theorem 1. Let ψ_i^* denote the optimal policy for agents of type i when perceiving success probabilities $p_B > 0$ and $p_S > 0$ (as predetermined in the Theorem to be proved). Define the money holdings m_u^i according to Theorem 1.

Fix a type i . Given the policy ψ_i^* , agents' characteristics U^i , p^i , and the success probabilities $p_B > 0$ and $p_S > 0$, the rationing scheme (detailed at the end of section 2) defines a Markov chain on the set of money holdings $X^i := \{0, \dots, m_{\min U^i}\}$.

Let us determine the transition probabilities $P_i(m, m')$, $m, m' \in X^i$, evoking the result of Theorem 1. In each trade exactly one unit of money changes hands, therefore $P_i(m, m')$ can be strictly positive if and only if $|m - m'| \leq 1$. One finds $P_i(0, 1) = p_S$ (because all agents without money choose to sell), $P_i(m_{\min U^i}, m_{\min U^i} - 1) = p_B$ (because $m_{\min U^i}$ is the maximal money holdings), and for all $0 < m < m_{\min U^i}$, $P_i(m, m-1) = p_B \sum_{u \in U^i} p_u^i \mathbf{1}_{m \geq m_u^i}(m)$ and $P_i(m, m+1) = p_S \sum_{u \in U^i} p_u^i \mathbf{1}_{m < m_u^i}(m)$ (because an agent demands (supplies) the good when u occurs and he holds more than or equal to (less than) m_u units of money). All of these transition probabilities are strict positive because $p_B > 0$, $p_S > 0$, $p_u^i > 0$, and $m_{\max U^i} = 1$.

Summarizing these findings, the Markov chain on X^i turns out to be irreducible. Denote the corresponding unique stationary distribution by $\tilde{\mu}^i$. We can now define a probability measure μ^* on $I \times \mathbb{N}$ by $\mu_i^*(m) := \lambda^i \tilde{\mu}^i(m)$, $i \in I$, and denote by $M^* := \sum_{i \in I, m \in \mathbb{N}} m \mu_i^*(m)$ the corresponding (real) quantity of money.

It remains to show that $(\psi^*, \mu^*, p_B^*, p_S^*, M^*)$ is a stationary equilibrium, where $p_B^* := p_B$ and $p_S^* := p_S$. We check each condition in Definition 1 in turn. First, the stationary policy ψ^* is optimal by construction. Second, μ^* is stationary and the marginal distribution over types is $\sum_m \mu_i^*(m) = \lambda^i \sum_m \tilde{\mu}^i(m) = \lambda^i$ also by construction. Third, we have to show that the actual success probabilities, cf. (4), are equal to the prescribed ones p_B^* and p_S^* . Let us consider the case $p_S^* = 1$ (the case $p_B^* = 1$ is analogous). Condition (iii) of Definition 1 is satisfied if and only if

$$p_B^* \sum_{m \in \mathbb{N}, i \in I, u \in U^i} \mu_i^*(m) p_u^i \mathbf{1}_{\psi_i^*(m, u)}(B) = \sum_{m \in \mathbb{N}, i \in I, u \in U^i} \mu_i^*(m) p_u^i \mathbf{1}_{\psi_i^*(m, u)}(S) \quad (5)$$

Let us recall stationarity of $\tilde{\mu}^i$, which implies ($\underline{m}^i := m_{\min U^i}$)

$$p_B^* \left(\tilde{\mu}^i(\underline{m}^i) + \sum_{m < \underline{m}^i} \tilde{\mu}^i(m) \sum_{u \in U^i} p_u^i \mathbf{1}_{m \geq m_u^i}(m) \right) = \tilde{\mu}^i(0) + \sum_{m < \underline{m}^i} \tilde{\mu}^i(m) \sum_{u \in U^i} p_u^i \mathbf{1}_{m < m_u^i}(m)$$

That is, for each fixed type of agent, the amount of agents whose money holdings decrease by one unit (due to actually buying the good) is equal to the amount of agents whose money holdings increase by one (due to actually selling the good). This holds by stationarity of $\tilde{\mu}^i$ for the corresponding Markov chain on X^i .

Taking into account that $\psi_i^*(m, u) = B$ if and only if $m \geq m_u^i$ (and $\psi_i^*(m, u) = S$ if and only if $m < m_u^i$), multiplying both sides of the last equation by λ^i and sum over $i \in I$, we obtain (5).

Finally, condition (iv) of Definition 1 holds by our definition of M^* . The equilibrium is monetary by Lemma 1. \square

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